NOTE ON THE FORMULATION OF FINITE DIFFERENCE EQUATIONS INCORPORATING A MAP SCALE FACTOR

FREDERICK G. SHUMAN and JOHN D. STACKPOLE

National Meteorological Center, Weather Bureau, ESSA, Washington, D.C.

ABSTRACT

Numerical experimentation with various finite difference formulations of a particular set of differential equations incorporating a map scale factor indicates that the stability of the calculations is as dependent upon the manner in which the map factor is introduced as the form in which the dynamic terms of the equations are written.

1. INTRODUCTION

Some years ago Shuman [1] studied the nonlinear, computational instability of various finite difference formulations of the equations describing external gravity waves in a longitudinally bounded region of homogeneous incompressible fluid of infinite lateral extent. His study was an adjunct to a more general study of numerical experimentation with primitive equations and had the purpose of suggesting what finite difference formulation would be most appropriate for the full set of hemispheric prediction equations. In figure 1, Plates 1 and 2 taken directly from Shuman's paper, show: on Plate 1, the differential equations, finite difference net and formulations, the boundary conditions, and a trigonometric formulation for the waves; on Plate 2, the various finite difference formulations investigated by Shuman. His study indicated that only the semi-momentum and filtered factor forms were sufficiently well behaved to merit further consideration and adaptation to the full three dimensional prediction equations.

The full set of equations, being hemispheric, include a map factor appropriate to the projection employed. The mode of inclusion of this factor into the finite difference equations was not given any particular consideration other than to maintain the three dimensional analog to the one dimensional form as much as possible. Recent difficulties in integrating the hemispheric equations for extended times have led to the suspicion that further, more detailed, consideration should be given to the precise method of inclusion of the map factor. To this end the external gravity wave equations were rewritten with a map factor included and a number of finite difference formulations investigated numerically.

2. FINITE DIFFERENCE FORMULATIONS

On a projection with map factor m the differential equations of Plate 1 (fig. 1) become

$$\frac{\partial u}{\partial t}+m\frac{\partial}{\partial x}\left(\frac{u^2}{2}+gh\right)=0$$

$$\frac{\partial h}{\partial t}+m\frac{\partial}{\partial x}(uh)=0$$

and the various finite difference formulations which were studied are exhibited below:

Semi-Momentum I:

$$\overline{u_t}^t+\overline{\overline{m}^x\,\overline{u}^x u_x}^x+g\overline{\overline{m}^x h_x}^x=0$$

$$\overline{h_t}^t+\overline{\overline{m}^x\,\overline{u}^x h_x}^x+\overline{\overline{m}^x\,\overline{h}^x u_x}^x=0$$

Semi-Momentum II:

$$\overline{u_t}^t+\overline{m}^{xx}\,\overline{\overline{u}^x u_x}^x+g\overline{m}^{xx}\,\overline{h_x}^x=0$$

$$\overline{h_t}^t+\overline{m}^{xx}\,\overline{(hu)_x}^x=0$$

Filtered Factor I:

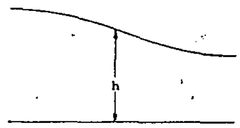
$$\overline{u_t}^t+\overline{mu}^{xx}\,\overline{u_x}^x+g\overline{\overline{m}^x h_x}^x=0$$

$$\overline{h_t}^t+\overline{mu}^{xx}\,\overline{h_x}^x+\overline{mh}^{xx}\,\overline{u_x}^x=0$$

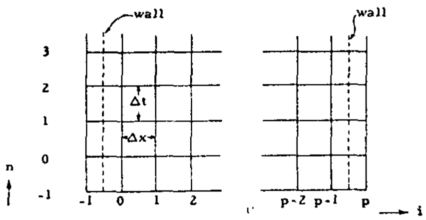
DIFFERENTIAL EQUATIONS:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial h}{\partial x}=0$$

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+h\frac{\partial u}{\partial x}=0$$



FINITE-DIFFERENCE NET:



$$i=\frac{x}{\Delta x} \qquad n=\frac{t}{\Delta t}$$

DEFINE:

$$\overline{u}^x=\frac{1}{2}(u_{i+1/2}+u_{i-1/2})$$

$$u_x=\frac{1}{\Delta x}(u_{i+1/2}-u_{i-1/2})$$

$$\overline{u}^t=\frac{1}{2}(u_{n+1/2}+u_{n-1/2})$$

$$u_t=\frac{1}{\Delta t}(u_{n+1/2}-u_{n-1/2})$$

BOUNDARY CONDITIONS:

$$\overline{u}^x=h_x=0 \quad \text{at} \quad i=-\frac{1}{2}\,,\quad p-\frac{1}{2}$$

THESE SUGGEST TRIGONOMETRIC ANALYSES IN THE FORM:

$$u=\sum_{k=1}^{p} U_k \sin\frac{\pi k(i+1/2)}{p}$$

$$h=\sum_{k=0}^{p-1} H_k \cos\frac{\pi k(i+1/2)}{p}$$

Plate 1

MOMENTUM FORM

$$(\overline{hu})^t_t+(\overline{huu})^x_x+gh\overline{h}^x_x=0$$

or

$$(\overline{hu})^t_t+(\overline{huu})^x_x+\tfrac{1}{2}g(\overline{hh})^x_x=0$$

$$\overline{h}^t_t+(\overline{hu})^x_x=0$$

SEMI-MOMENTUM FORM

$$\overline{u}^t_t+\tfrac{1}{2}(\overline{uu})^x_x+g\overline{h}^x_x=0$$

$$\overline{h}^t_t+(\overline{hu})^x_x=0$$

ADVECTIVE FORM

$$\overline{u}^t_t+u\overline{u}^x_x+g\overline{h}^x_x=0$$

$$\overline{h}^t_t+u\overline{h}^x_x+h\overline{u}^x_x=0$$

FILTERED FACTOR FORM

$$\overline{u}^t_t+\overline{u}^{xx}\overline{u}^x_x+g\overline{h}^x_x=0$$

$$\overline{h}^t_t+\overline{u}^{xx}\overline{h}^x_x+\overline{h}^{xx}\overline{u}^x_x=0$$

THE FILTERED FACTOR FORM CAN BE WRITTEN

$$\overline{u}^t_t+\tfrac{1}{2}(\overline{u}^x\overline{u}^x)_x+g\overline{h}^x_x=0$$

$$\overline{h}^t_t+(\overline{h}^x\overline{u}^x)_x=0$$

Plate 2

FIGURE 1.—Plates 1 and 2, reproduced from [1].

Filtered Factor II:

$$\overline{u}^t_t+\overline{m}^{xx}\,\overline{u}^{xx}\,\overline{u}^x_x+g\overline{m}^{xx}\,\overline{h}^x_x=0$$

$$\overline{h}^t_t+\overline{m}^{xx}\,\overline{u}^{xx}\,\overline{h}^x_x+\overline{m}^{xx}\,\overline{h}^{xx}\,\overline{u}^x_x=0$$

Smagorinsky-Momentum:

$$\overline{(hu)}^t_t+\overline{m}^{xx}(\overline{hu}^x\,\overline{u}^x)_x+\frac{g}{2}\,\overline{m}^{xx}\,\overline{(h^2)}^x_x=0$$

$$\overline{h}^t_t+\overline{m}^{xx}\,\overline{(hu)}^x_x=0$$

Arakawa-Momentum:

$$\overline{(hu)}^t_t+\overline{m}^{xx}(\overline{h}^x\,\overline{u}^x\,\overline{u}^x)_x+\frac{g}{2}\,\overline{m}^{xx}\,\overline{(h^2)}^x_x=0$$

$$\overline{h}^t_t+\overline{m}^{xx}(\overline{h}^x\,\overline{u}^x)_x=0$$

The Semi-Momentum I form is the one most closely resemblingt he manner in which the three dimensional equations were formulated while the Semi-Momentum II form represents a sort of simplification of the previous one. The two filtered factor forms were included for completeness. Finally, the Smagorinsky-Momentum [2] and Arakawa-Momentum [3, 4] forms were added to the collection for study as these are the forms used by other investigators doing long time integrations of atmospheric models.

3. NUMERICAL EXPERIMENTATION

These various formulations were programed to run on the National Meteorological Center machine with various initial conditions similar to those used by Shuman, i.e., the initial height of the fluid a constant equal to 25,000 ft., time increment Δt equal to 10 min., space increment Δx equal to 381 km., and p the number of grid points between the bounding walls equal to 24. For the calculations described here, the map factor was that for a polar stereographic projection true at 60°N.:

$$m=\frac{1+\sin 60°}{1+\sin \phi}$$

with ϕ the latitude; this was specified for the grid points such that the points outside the wall (i.e., $p=-1$ and 24 in Plate 1 of fig. 1) were projected to fall on the Pole and the Equator. In this context then the x coordinate of the equations may be considered as the north-south coordinate on a nonrotating earth. The initial time step is an uncentered forward difference and the initial velocities are read in terms of a table of 24 Fourier components U_k and these are synthesized according to the formula of Plate 1 (without any map factor) to give wind speeds u at the grid points. Four types of wind speed input data were used: "High Energy Wave 1" in which $U_1=54.6$

m. sec.$^{-1}$ and all the other Fourier components were zero; "½ High Energy Wave 1," $U_1=25.0$ m. sec.$^{-1}$ and $U_k=0$, $k=2, 24$; "White Noise" in which all the U_k's equaled 2.18 m. sec.$^{-1}$; and "White Noise less no. 24," the same as "White Noise" except that $U_{24}=0$.

Once begun the integrations were programed to continue until one of the following occurred: a) 50,000 time steps were achieved (this is roughly 347 days, which would seem sufficiently long); b) computational instability arose as evidenced by (1) the depth becoming negative (under this condition the *differential* equations are unstable) or (2) the height or wind speed exceed 100 times their initial or RMS values respectively.

Perhaps the most efficient way of studying and comparing the integrations with different initial conditions is in terms of the energetics of the flows. To this end the program computed the available potential energy (per unit mass)

$$P=\tfrac{1}{2}g\frac{\Delta x}{m}\sum_{i=0}^{p}(h_i-H_0)^2$$

and kinetic energy (per unit mass)

$$K\cong \tfrac{1}{2}H_o\frac{\Delta x}{m}\left(\sum_{i=0}^{p}u_i^2-\tfrac{1}{4}pU_p^2\right)+\tfrac{1}{2}\frac{\Delta x}{m}\sum_{i=0}^{p}(h_i-H_0)u_i^2$$

(the formulae are from Shuman [1]) in which g is the acceleration of gravity; p is the number of grid points within the walls; h_i is the fluid depth at the ith grid point; H_o is the spectral amplitude of wave number zero, i.e., the initial mean height of the fluid; u_i is the wind speed at the ith grid point; and U_p is the velocity spectral amplitude for wave number p (subject to truncation error and hence dealt with separately). The machine was programed to print out a graphical display of the energy record and figure 2 (prepared by the computer) is such a record for the first 200 steps ($\sim$ 33 hr.) of an integration using the Semi-Momentum II form for the finite differencing and the "High Energy Wave no. 1" initial conditions. The total energy column is the algebraic sum of P and K and on the graph is indicated by T. The total energy remains admirably constant during the period presented and a straight forward and initially almost complete conversion of kinetic to potential energy and back is seen as a consequence of the fluid sloshing back and forth. Evidence of the nonlinear interactions inducing modes of flow other than the initial conditions may be seen in the slight jaggedness in the P and K records near step no. 170. Inspection of the corresponding records for the other finite difference formulations with the same initial conditions shows them to be much the same for the same period as are the sets of records for the other initial conditions.

The close similarity does not remain for long however. Figure 3 shows the continuation of the energy records for another 200 steps for the Semi-Momentum I and II cases and, in contrast, the Arakawa-Momentum formulation. Again, the Semi-Momentum II case shows quite satisfactory conservation of the total energy while Semi-Momentum I does not nor does the Arakawa-Momentum case. (The energy graphs were printed modulo the width

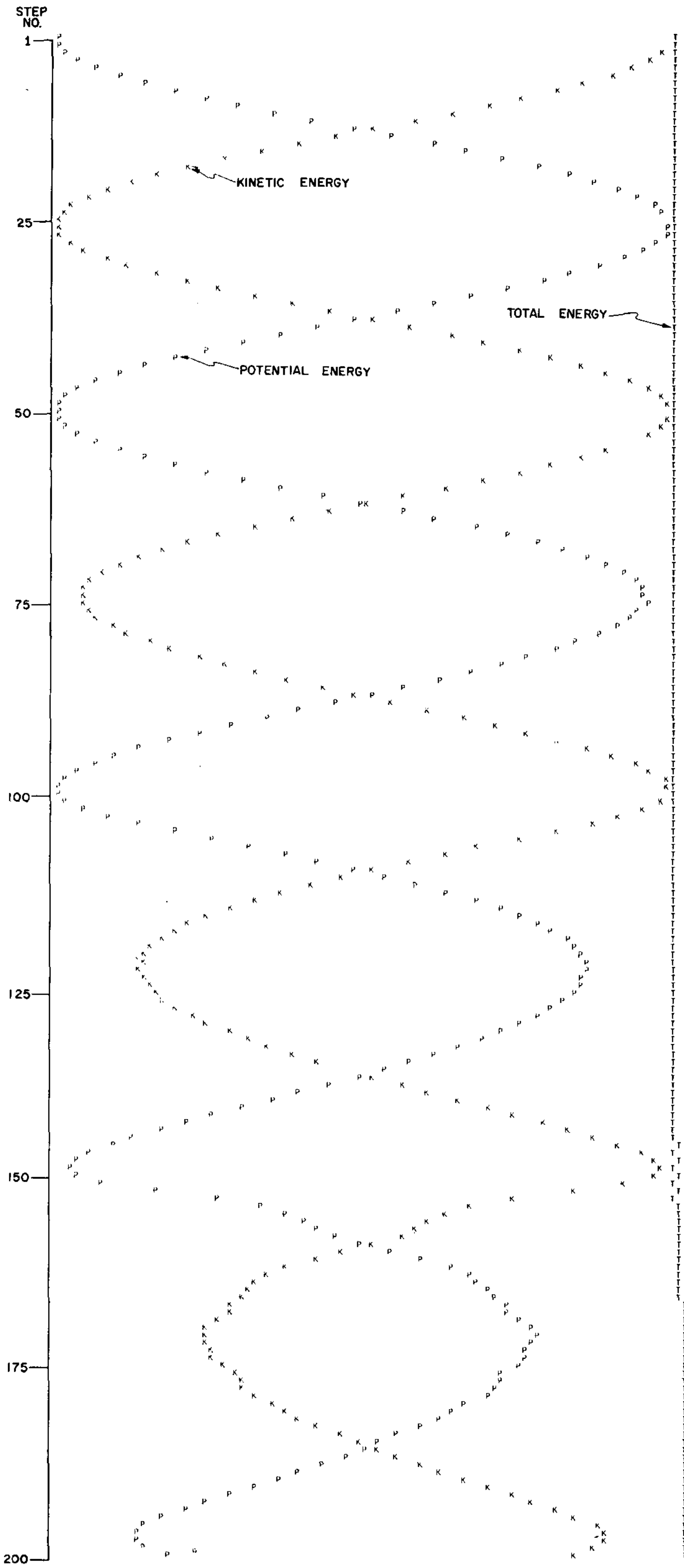


FIGURE 2—Relative energetics on linear scale, steps 1–200, Semi-Momentum II form. Initial: High Energy Wave 1.

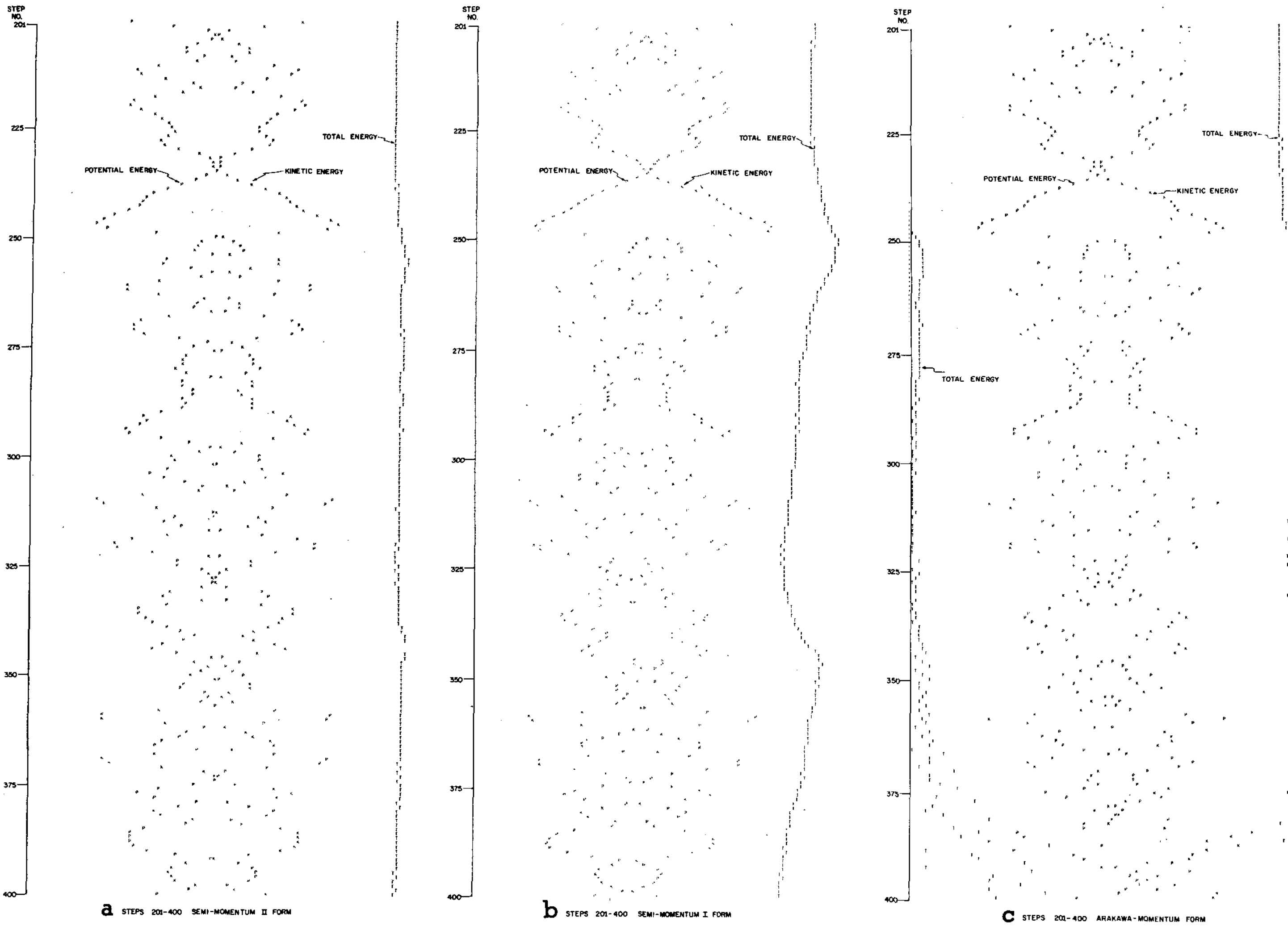


FIGURE 3.—Relative energetics on linear scale, steps 201–400. Initial: High Energy Wave 1. (a) Semi-Momentum II form. (b) Semi-Momentum I form. (c) Arakawa-Momentum form.

of the printed page so the appearance of the "T's" on the left should be interpreted as a continuation of the record off the right-hand side of the figure.) The behavior of the Arakawa-Momentum case was sufficiently violent that the flow violated the negative depth criterion at step No. 477 and the integration was terminated; the Semi-Momentum I flow continued to step No. 6,032 before violating the same criterion while the Semi-Momentum II case continued without difficulty through 50,000 steps. The Smagorinsky-Momentum form behaved similarly to the Arakawa-Momentum calculations although it was somewhat more viable, *vide* table 1.

It cannot be said that the 50,000-step forecast had any physical meaning. By the time step 2,000 was reached the Semi-Momentum II case was already showing considerable alternation in the total (and component) energies between odd and even time steps. This phenomenon can be seen also in the later stages of the Arakawa-Momentum calculation of figure 3(c). By the time 50,000 steps were reached the energy graphs were completely incoherent with total energy variations of a factor of two or so from one time step to the next. However, and this is the more significant point, the average energy had increased by no more than 15 percent during the integration indicating the inherent stability of this particular finite difference formulation.

TABLE 1.—*Summary of results*

Formulation	Initial conditions			
	High energy no. 1	½ High energy no. 1	White noise	White noise less no. 24
Semi-Momentum I	Dry @ 6,032	Dry @ 22,404	Dry @ 17,986	Dry @ 14,304
Semi-Momentum II	Stable thru 50,000	Stable thru 50,000	Stable thru 50,000	Stable thru 50,000
Filtered Factor I	Dry @ 3,511	Dry @ 11,377	$u > u_{max}$ @ 28,111 2–3 orders of magnitude energy increases	$u > u_{max}$ @ 27,015 2–3 orders of magnitude energy increase
Filtered Factor II	Dry @ 6,784	Dry @ 18,373	Stable thru 50,000 2–3 orders of magnitude energy increase	Stable thru 50,000 2–3 orders of magnitude energy increase
Smagorinsky-Momemtum	Dry @ 1,411	Dry @ 6,275	$u > u_{max}$ @ 11,006	$u > u_{max}$ @ 11,340
Arakawa-Momentum	Dry @ 477	Dry @ 1,970	$u > u_{max}$ @ 297	$u > u_{max}$ @ 289

Table 1 is a summary of the ultimate behavior of the various combinations of finite-difference formulations and initial conditions investigated. "Dry" indicates that the negative depth criterion was violated at the step number indicated. The noted large increase of energy for the two filtered factor forms that either continued through or almost reached 50,000 steps is consistent with the observation by Shuman that his filtered factor form showed an increase in energy during the relatively short integration times he considered. It is obvious that the Semi-Momentum II formulation should be the one chosen to serve as guidance in the formulation of three dimensional finite-difference equations, in which gravity wave motion may be of importance in the absence of other constraints upon the flow, and long time integrations are contemplated.

It is realized that conservation of energy is not the ultimate requirement for success in finite-difference calculations but it is necessary, and we have used it here as a guide which we trust will prove helpful. The problem of the separation of odd and even time steps is one of obvious importance, Arakawa [4] has dealt with it by making a centered forward time step at regular intervals. We felt this to be of considerable interest but not germane to our present effort which was the investigation of the effects of various space differencing methods. Finally, Arakawa mentions that in his most recent work he is making use of another space differencing method which was not available to us at the times the above work was done.

4. COMMENTARY

In the absence of any well-developed theory of nonlinear and finite difference stability one is hard pressed to offer any complete explanation of the behavior of the various formulations. A couple of observations are possible which may serve in a practical manner as partial guidance for the finite difference formulations of more complete equations.

In statistical terms it is very familiar that the covariance of two quantities x and y may be written as

$$\overline{x'y'}=\overline{xy}-\bar{x}\,\bar{y}$$

where the overbar indicates a sample or population mean and the prime denotes the departure of an individual member from that mean. Now considering the overbars of the finite difference notation as indicating averages (over a sample of only two elements to be sure) it is easy to see that the only difference between the Semi-Momentum I and II forms is that the latter neglects the (local, two grid point) covariances of $\overline{m}^x$ with $\overline{u}^x$, u_x, h_x and $(hu)_x$, while the former implicity includes them. That these covariances are small is evidenced by the similarity of the integrations during their early stages alluded to previously; that they are of major importance in a cumulative manner is, of course, evidenced by the ultimate fate of the computations. Suppressing the covariances between the map factor term and the dynamic terms resulted in stabilization of the integrations.

In addition, one notes that the Filtered Factor II equations disregard similar covariance terms between the map factor and dynamic quantities which are included in the Filtered Factor I equations. A glance at table 1 shows that indeed the former set of equations behaves in a somewhat more stable manner than the latter.

Comparison of the Smagorinsky- and Arakawa-Momentum formulations in these same terms points up a somewhat different result. In both these formulations the map factor-dynamic covariance is suppressed and the difference lies in the advective terms. In the Arakawa formulation the h-u local covariances are suppressed while they are included in the Smagorinsky form. Table 1 indicates that the latter formulation is the better behaved. Comparison of the results for Semi-Momentum II and Filtered Factor II points to the same conclusion: that the stability of the integrations is enhanced by the inclusion of the covariances between the dynamic terms themselves. The comparison between Semi-Momentum I and Filtered Factor I is not so clear cut but this can be perhaps accounted for by noting the destabilizing effect of the map factor covariances present in both formulations.

If we may draw a general rule to use in generalizing from these results, it would appear quite obviously that inclusion of dynamic interactions as completely as possible, i.e., the covariances between the u's and h's is appropriate and desirable while the interactions between such artificial parameters as the map factor and the dynamic quantities should be suppressed as much as possible consistent with the original differential equations.

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